Optimization Problems

Solutions should show all of your work, not just a single final answer.

1. We want to find the points on $y = x^2$ that are closest to $(0, 3)$.

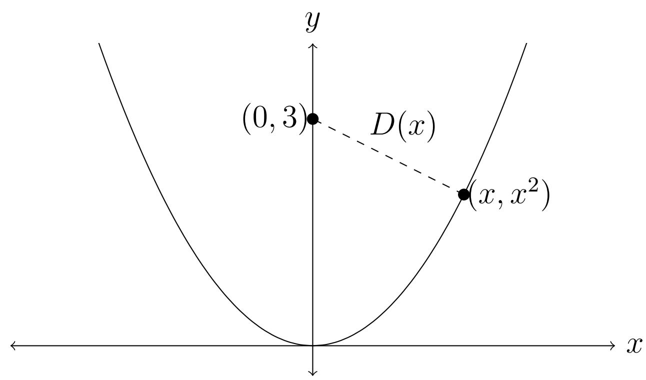

Figure 1: Distance to $(0, 3)$ on $y = x^2$.

   (a) For each point (x, x^2) on the parabola, find a formula for its distance to $(0, 3)$. Call this distance $D(x)$. (See Figure 1.) Let $f(x) = D(x)^2$, which is the *squared distance* between (x, x^2) and $(0, 3)$. Finding where $D(x)$ is minimal is the same as finding where $f(x)$ is minimal.

   (b) Find all x where $f(x)$ has a local minimum. The points (x, x^2) for such x are the closest points to $(0, 3)$ on $y = x^2$.

(worksheet continues on next page)

2. Three line segments of length 1 are joined together at endpoints to form a base and the legs of an isosceles trapezoid, as in Figure 2. Let θ in $(0, \pi/2)$ be the common angle measurement between the legs and the line passing through the base of length 1. We want to find the angle θ that maximizes the area of the trapezoid.

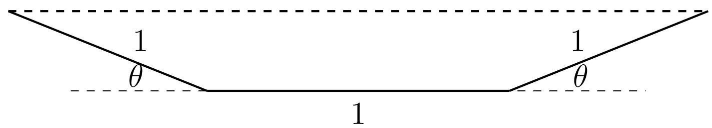

Figure 2: An isosceles trapezoid with base and legs of length 1.

(a) Compute the area $A(\theta)$ of the trapezoid. The general area formula for a trapezoid is $\frac{1}{2}h(b_1 + b_2)$, where h is the height and b_1 and b_2 are the lengths of the bases. (Hint: Break up the trapezoid into a rectangle with two right triangles at both ends. Use trigonometry to compute the height and the length of the longer base in terms of θ.)

(b) Find all solutions to $A'(\theta) = 0$ with $0 < \theta < \pi/2$. (The answer is *not* $\pi/4 = 45^o$.)

(c) Verify that the area $A(\theta)$ is a maximum, not a minimum, at the angle found in part (b) and compute this maximum area.

3. A closed box (top, bottom, and all four sides) needs to be constructed to have a volume of 9 m^3 and a base whose width is twice its length. Use calculus to determine the dimensions (length, width, height) of such a box that uses the least amount of material.