More Applications of Derivatives

Name: ____________________ Section No: __________

Derivatives and Graphs

1. For the following functions, use first and second derivatives to determine (i) all points x where $f(x)$ is a local maximum or minimum value, (ii) all open intervals where $f(x)$ is increasing, decreasing, concave up, and concave down. Give all answers **exactly**, not as numerical approximations.

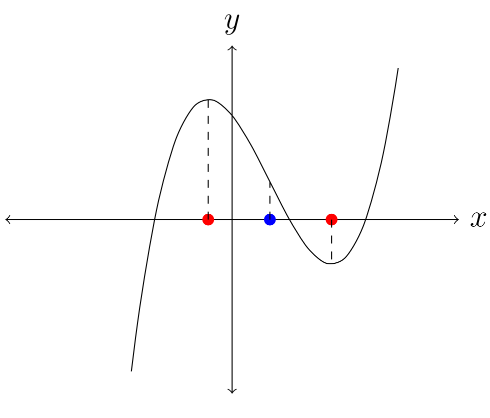
   (a) $f(x) = x^5 - 2x^3$

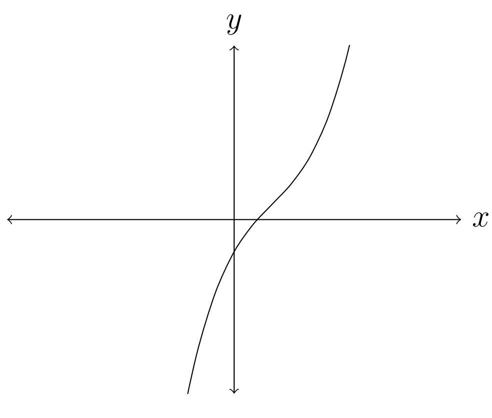
   (b) $f(x) = x + \sin x$ for $-2\pi \le x \le 2\pi$

   (c) $f(x) = e^{-x} - e^{-3x}$

2. Let $f(x) = x^{100} + (100 - x)^{100}$. Determine where $f(x)$ is increasing and decreasing for $0 \le x \le 100$ and use this information to decide which of $33^{100} + 67^{100}$ or $41^{100} + 59^{100}$ is larger.

3. Below are graphs of two cubic polynomials. The left one has two critical points, in red, and one inflection point, in blue. The one on the right has no critical points.





   In the graph on the left, the inflection point is the average of the two critical points: it lies exactly in the middle between them. Show this is a general phenomenon: for every cubic polynomial $f(x) = ax^3 + bx^2 + cx + d$ that has two critical points, say r and s, there is one inflection point and it is $(r + s)/2$. (Hint: critical points are roots of $f'(x)$, a quadratic polynomial. Use the quadratic formula.)

L'Hospital's Rule

4. Describe each limit's indeterminate form and then compute the limit. If l'Hospital's Rule is needed more than once, try to simplify the expression before applying it.

   (a) $\lim_{x \to 0} \frac{(x+1)^{11} - 11x - 1}{x^2}$

   (b) $\lim_{x \to 0} \frac{\sin(3x)}{e^{9x} - e^{2x}}$

   (c) $\lim_{x \to \infty} \frac{\ln(2015x^2 + 1)}{\ln x}$

   (d) $\lim_{x \to 0^+} \frac{\ln(\sin(2x))}{\ln(\sin(3x))}$

   (e) $\lim_{x \to 0} \frac{\ln(\cos(2x))}{\ln(\cos(3x))}$

5. Try to use l'Hospital's Rule on $\lim_{x \to \infty} \frac{x}{\sqrt{x^2+1}}$ and explain what goes wrong. Then evaluate this limit using earlier methods from the course.

Optimization

6. We want to find the points on $y = x^2$ that are closest to $(0, 3)$.

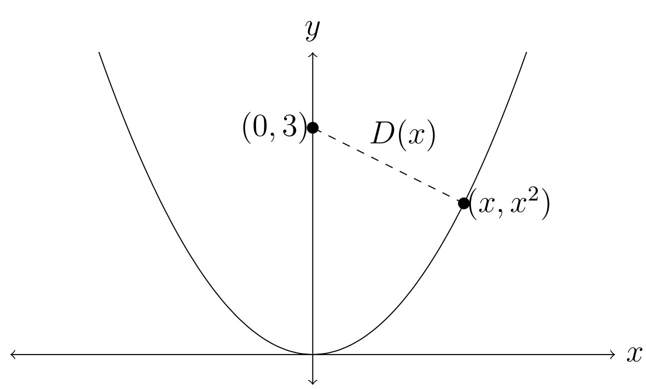


Figure 1: Distance to $(0, 3)$ on $y = x^2$.

   (a) For each point (x, x^2) on the parabola, find a formula for its distance to $(0, 3)$. Call this distance $D(x)$. (See Figure 1.)

(b) Let $f(x) = D(x)^2$, which is the *squared distance* between (x, x^2) and $(0, 3)$. Show $f'(x)$ is 0, positive, or negative exactly when $D'(x)$ is 0, positive, or negative. So finding out where $D(x)$ is minimal is the same as finding out where $f(x)$ is minimal.

(c) Find all x where $f'(x) = 0$ and determine among such x where $f(x)$ (and thus $D(x) = \sqrt{f(x)}$) has a local minimum value. Explain why $f(x)$ has an absolute minimum value at such x. The points (x, x^2) for such x are the closest points to $(0, 3)$ on $y = x^2$.

(d) Where does $f'(x) = 0$ and $f(x)$ *not* have a local minimum value? Is there something special about the distance of such points (x, x^2) to the point $(0, 3)$?

7. Three line segments of length 1 are joined together at endpoints to form a base and the legs of an isosceles trapezoid, as in Figure 2. Let θ in $(0, \pi/2)$ be the common angle measurement between the legs and the line passing through the base of length 1. We want to find the angle θ that maximizes the area of the trapezoid.

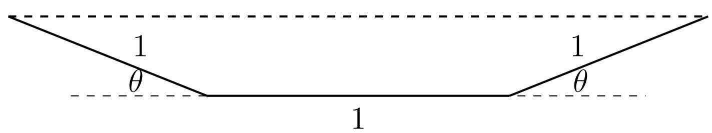


Figure 2: An isosceles trapezoid with base and legs of length 1.

(a) Compute the area $A(\theta)$ of the trapezoid. The general area formula for a trapezoid is $\frac{1}{2}h(b_1 + b_2)$, where h is the height and b_1 and b_2 are the lengths of the bases. (Hint: Break up the trapezoid into a rectangle with two right triangles at both ends. Use trigonometry to compute the height and the length of the longer base in terms of θ.)

(b) Find all solutions to $A'(\theta) = 0$ with $0 < \theta < \pi/2$. (The answer is *not* $\pi/4 = 45^o$.)

(c) Verify that the area $A(\theta)$ is a maximum, not a minimum, at the angle found in part b and compute this maximum area.