Section 6.5: The Fundamental Theorem of Calculus
Section Objectives:

- Know the following Properties of Definite Integrals:
  - $\int_a^a f(x)\,dx = 0$
  - $\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$ where k is a constant
  - $\int_a^b f(x) \pm g(x)\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$
  - $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$ where $a < c < b$.
  - $\int_b^a f(x)\,dx = \int_a^b f(x)\,dx$
- Know the statement of the Fundamental Theorem of Calculus, Part 1, i.e if $f(x)$ is continuous on $[a, b]$:
$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$
.
- Know the statement of the Fundamental Theorem of Calculus, Part 2, i.e. if $f(x)$ is continuous on $[a, b]$:
$$\int_a^b f(x)\,dx = F(b) - F(a)$$
where $F(x)$ is any antiderivative of $f(x)$.
- Use FTC 2 to evaluate definite integrals.
- Use definite integrals to find the overall change in a function from its rate of change (i.e. change in cost from marginal cost, number sold from rate of sale, etc.).

Practice Problems

1. Assume $\int_0^3 f(x)\,dx = 5$, $\int_3^4 f(x)\,dx = 2$ and $\int_0^3 g(x)\,dx = 4$. Evaluate the following:

   (a) $\int_0^3 4f(x)\,dx$ $= 4\int_0^3 f(x)\,dx = 4 \cdot 5 = \boxed{20}$

   (b) $\int_0^3 f(x) - 2g(x)\,dx$ $= \int_0^3 f(x)\,dx - 2\int_0^3 g(x)\,dx = 5 - 2(4) = \boxed{-3}$

   (c) $\int_0^4 f(x)\,dx$ $= \int_0^3 f(x)\,dx + \int_3^4 f(x)\,dx = 5 + 2 = \boxed{7}$

   (d) $\int_3^0 4g(x)\,dx$ $= -4\int_0^3 g(x)\,dx = -4(4) = \boxed{-16}$

2. Let $f(x) = \int_3^x e^{3t}\,dt$. Find $f(3)$ and $f'(x)$.

$f(3) = \int_3^3 e^{3t}\,dt = 0$ $\quad f'(x) = e^{3x}$

(plug in x to function in integral)

3. Evaluate the following definite integral using FTC 2.

(a) $\int_0^2 x^2\,dx$ $= \frac{x^3}{3}\Big|_0^2 = \frac{8}{3} - 0 = \boxed{8/3}$

(b) $\int_1^4 \frac{3}{x^2}\,dx$ $= \int_1^4 3x^{-2}\,dx = \frac{3x^{-1}}{-1}\Big|_1^4 = -\frac{3}{x}\Big|_1^4$

$= -\frac{3}{4} + \frac{3}{1} = \boxed{2.25}$

(c) $\int_2^5 \frac{e^x}{1+e^x}\,dx$ $\quad u = e^x + 1$, $du = e^x\,dx$

$\int_{x=2}^{x=5} \frac{1}{u}\,du = \ln|u|\Big|_{x=2}^{x=5} = \ln(e^x+1)\Big|_2^5 = \boxed{\ln(e^5+1) - \ln(e^2+1)}$

(d) $\int_2^7 \frac{2x}{3x^2+1}\,dx$ $\quad u = 3x^2+1$, $du = 6x\,dx$, $\frac{du}{3} = 2x\,dx$

$\int_{x=2}^{x=7} \frac{1}{u}\frac{du}{3} = \frac{1}{3}\ln|u|\Big|_{x=2}^{x=7} = \frac{1}{3}\ln(3x^2+1)\Big|_{x=2}^{x=7} = \boxed{\frac{1}{3}\ln(148) - \frac{1}{3}\ln(13)}$

4. Mustafa finds that the rate of sales of an item is given by $S'(t) = -3t^2 + 36t$ where t is number of weeks after an advertising campaign. How many items are sold during the third week?

1st week $t = 0$ to 1

2nd 1 to 2

3rd 2 to 3

$\int_2^3 -3t^2 + 36t\,dt = -t^3 + 18t^2\Big|_2^3 = (-27 + 18 \cdot 9) - (-8 + 18 \cdot 4) = \boxed{71}$

More Practice from Textbook 6.5: You should do as many problems from each set (1-24, 35-38, 39-40, 41-57), as needed until you are comfortable with these techniques. 41-57 are good practice for application problems. (We are skipping average value of a function.)