Section 5.4: Additional Curve Sketching
Section Objectives: Sketch a graph of a function by

- Finding the domain of the function.
- Determining the symmetries of the functions
- Finding the horizontal and vertical asymptotes
- Finding the x and y intercepts.
- Using the first derivative to find where the function is increasing or decreasing and the relative extrema.
- Using the second derivative to find where the function is concave up or concave down and the inflections points.
- Putting all the above information together.

Practice Problems

1. Let $f(x) = \dfrac{x^2}{x^2-1}$. Use the steps listed above to sketch a graph of the function.

• Domain: $x^2-1 \neq 0$

$\Rightarrow (x-1)(x+1) \neq 0$

$x \neq 1, -1$

$$f(-x) = \frac{(-x)^2}{(-x)^2-1} = \frac{x^2}{x^2-1} = f(x)$$

• symmetric across y-axis (even function)

• Potential vertical asymptotes where

$x^2-1=0$ @ $x=\pm 1$

$$\lim_{x\to\pm 1} \frac{x^2}{x^2-1} = \frac{1}{0} = \text{"}\pm\infty\text{"}$$

as $x \to 1^+$ try 1.001 $f(1.001)=500.8$
so $+\infty$

as $x \to 1^-$ try .999 $f(.999) = -499.25$
so $-\infty$

as $x \to -1^-$ try -1.001 $f(-1.001) = 500.8$
so $+\infty$

as $x \to -1^+$ try $-.999$ $f(-.999) = -499.25$
so $+\infty$

VA @ $x = \pm 1$

• $$\lim_{x\to\pm\infty} \frac{x^2}{x^2-1} = \lim_{x\to\pm\infty} \frac{x^2/x^2}{\frac{x^2}{x^2}-\frac{1}{x^2}}$$

$$= \lim_{x\to\pm\infty} \frac{1}{1-1/x^2} = 1$$

HA @ $y=1$

• y-intercept → set $x=0$

$$y = \frac{0}{0-1} = 0$$

• x-intecept set $y=0$

$$0 = \frac{x^2}{x^2-1} \Rightarrow x^2 = 0$$

$\Rightarrow x = 0$

• $$f'(x) = \frac{(x^2-1)(2x)-x^2(2x}{(x^2-1)^2}$$

$$= \frac{-2x}{(x^2-1)^2}$$

$$f'(x) = \frac{-2x}{(x^2-1)^2} = 0$$ when $x=0$, undefined @ $x=\pm1$

(not critical points since not in domain but still consider).

f' + + + DNE + + + + 0 - - - - - - - DNE - - - - - - -

-1 0 1

f ↗ ↗ ↗ DNE ↗ ↗ ↗ ↘ ↘ ↘ ↘ DNE ↘ ↘ ↘ ↘

relative max at $x=0$

$$\left(0, \frac{0}{0-1}\right) = (0,0)$$

$$f''(x) = \frac{(x^2-1)^2(-2) + 2x(2(x^2-1))(2x)}{(x^2-1)^4} = -2(x^4-2x^2+1) + 8(x^4-x^2)$$

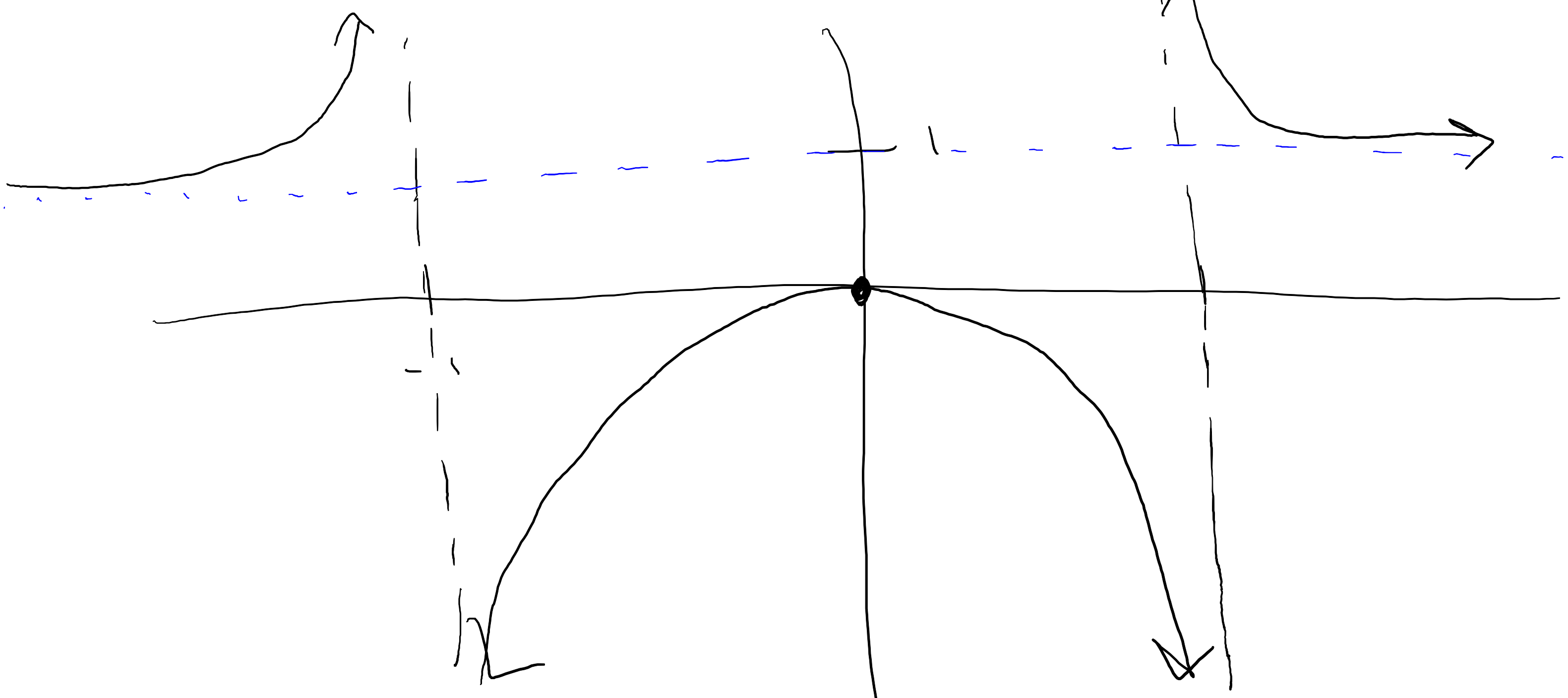
$$= \frac{-2x^4 + 4x^2 - 2 + 8x^4 - 8x^2}{(x^2-1)^4} = \frac{6x^4-4x^2-2}{(x^2-1)^4} = \frac{2(3x^4-2x^2-1)}{(x^2-1)^4}$$

$$= \frac{2(3x^2+1)(x^2-1)}{(x^2-1)^4} = \frac{2(3x^2+1)}{(x^2-1)^3}$$

since $3x^2+1 \neq 0$ $f''(x) \neq 0$, undefined at $x = \pm 1$

f'' + + + + + + + DNE - - - - - - DNE + + + + + +

f concave up -1 concave down +1 concave up



2. Let $f(x) = x\ln(x)$. Use the steps listed above to sketch a graph of the function.

$f(x) = x\ln x$

- Domain: $x > 0$ (since domain of $\ln(x)$ is $x > 0$).
- No symmetries since only defined $x > 0$
- $\ln(x)$ has VA @ $x = 0$. Check $\lim_{x \to 0} x\ln x$ (try small numbers).

  $= 0$, no VA.

- $\lim_{x \to \infty} x \cdot \ln x = \infty \cdot \infty = \infty$.

  So no HA.

- y intercept: let $x = 0$, not in domain

  no y-intercept

- x-intercept: let $y = 0$

  $0 = x \cdot \ln x$

  $\Rightarrow x = 0$ or $\ln x = 0$

  not in domain $\Rightarrow x = 1$

  so $(1,0)$ is only intercept.

- $f'(x) = x \cdot \frac{1}{x} + \ln x$

  $= 1 + \ln x$

$1 + \ln x = 0 \Rightarrow$

$-1 = \ln x$

$x = e^{-1} \approx .368$

f' DNE - - - - - - - - + + + + +

0 e⁻¹

f DNE ↘ ↘ ↘ ↘ ↘ ↗ ↗ ↗

$f(e^{-1}) = -\frac{1}{e}$

$\approx -.368$

$f''(x) = \frac{1}{x}$

never 0, undefined at $x = 0$

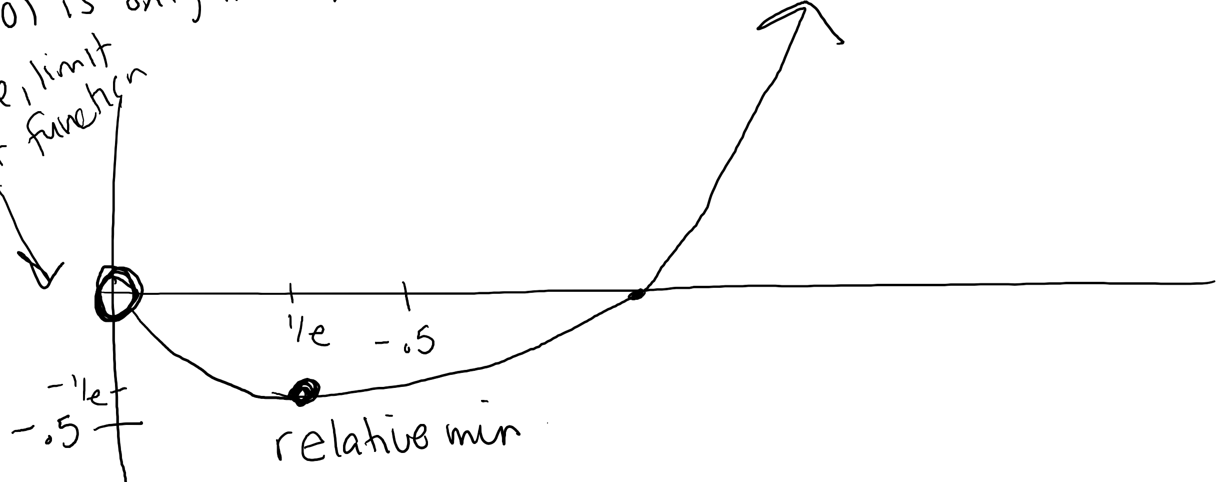
f'' DNE + + + + + + +

0

f concave up

no inflection points

open circle, limit exists but function undefined



relative min

3. Let $f(x) = x^4 - 2x^3 + x^2$. Use the steps listed above to sketch a graph of the function.

- Domain: all real numbers
- $f(-x) = (-x)^4 - 2(-x)^3 + (-x)^2$

  $= x^4 + 2x^3 + x^2$

  $\neq f(x)$ or $f(-x)$

  no symmetries
- HA: $\lim_{x \to \pm\infty} x^4 - 2x^3 + x^2$

  $= \lim_{x \to \pm\infty} x^4\left(1 - \frac{2}{x} + \frac{1}{x^2}\right)$

  $= \infty$. No HA
- VA: function always cont since polynomial so No VA
- x-intercepts: let $y = 0$

  $0 = x^2(x^2 - 2x + 1)$

  $x^2(x-1)(x-1)$

  $x = 0$, $x = 1$, $(0,0)$ & $(1,0)$
- y-intercepts let $x = 0$

  $\Rightarrow y = 0$

  $(0,0)$

$$f'(x) = 4x^3 - 6x^2 + 2x$$
$$= 2x(2x^2 - 3x + 1)$$
$$= 2x(2x - 1)(x - 1)$$

$f'(x) = 0$ when $x = 0, 1/2, 1$

f′	− · − · − ⊖		+ · − · − ⊕		+ · + · − ⊖		+ · + · + ⊕
f	↘ ↘	0	↗ ↗	$1/2$	↘ ↘	1	↗ ↗

relative min at $(0,0)$ & $(1,0)$

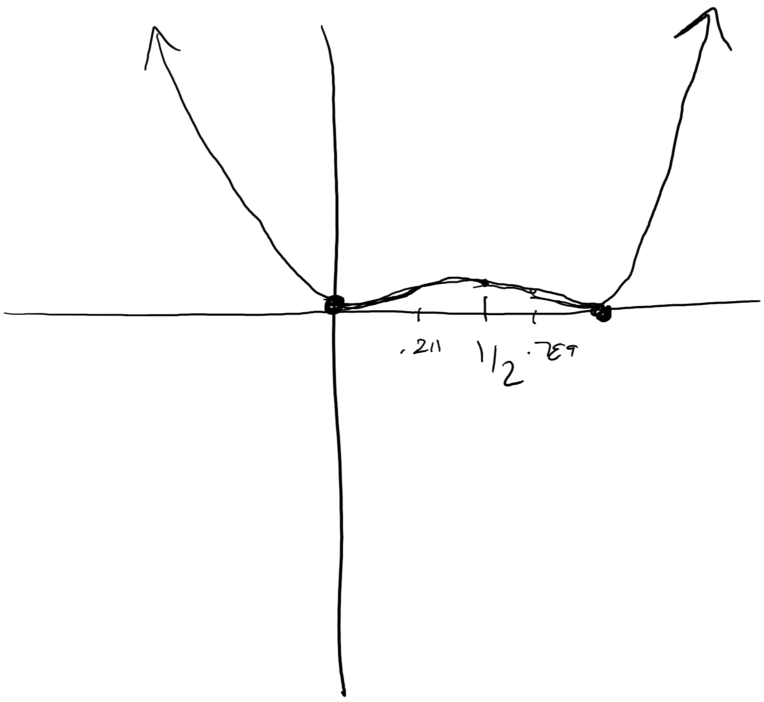
relative max at $(1/2, .0625)$

$$f''(x) = 12x^2 - 12x + 2$$
$$= 2(6x^2 - 6x + 1)$$

quadratic formula:

$$\frac{6 \pm \sqrt{36 - 4(6)}}{12}$$
$$= \frac{1}{2} \pm \frac{\sqrt{12}}{12} \approx .211 \text{ and } .789$$

f″	+ + + +	0	− − − − − − − − − −	0	+ + + + +
	concave up	.211	concave down	.789	concave up



More Practice from Textbook 5.4 You should do as many problems from each set (1-6, 7-12, 13-16, 17-24, 25-26, 37-42, 43-52), as needed until you are comfortable with these techniques. 37-42 are good practice for application problems.