Section 5.3: Limits at Infinity
Section Objectives:

- Know the definition of a horizontal asymptotes.
- Know how to algebraically find the limit at infinity of a rational function (by dividing by the highest power of x* on numerator and denominator).
- Know how to algebraically find the limit at infinity of a polynomial by considering the dominant (highest power) term.

*Note: when deciding what power to divide by, some people divide by the highest overall power of x (on either the numerator or denominator) while other people divide by the highest power present in the denominator. Both of these methods are valid and willl produce the proper result.

Practice Problems

1. In each of the following questions, plug in large values of x (or large negative values, where appropriate) to guess the limit. Then use algebraic techniques to evaluate the limit.

   (a) $\lim_{x\to\infty} \dfrac{3x^2+4x}{5x^2+3x}$

   Try $x = 1000$ Guess: $3/5$

   Divide by x^2 on top & bottom: $\lim_{x\to\infty} \dfrac{\frac{3x^2}{x^2} + \frac{4x}{x^2}}{\frac{5x^2}{x^2} + \frac{3x}{x^2}} = \lim_{x\to\infty} \dfrac{3 + \cancel{\frac{4}{x}}^{\,0}}{5 + \cancel{\frac{3}{x}}^{\,0}} = \boxed{3/5}$

   plug in $x=-1000$
   guess: 0

   (b) $\lim_{x\to-\infty} \dfrac{x-2}{x^2+4}$

   divide by x^2

   $\lim_{x\to-\infty} \dfrac{\frac{x}{x^2} - \frac{2}{x^2}}{1 + 4/x^2} = \lim_{x\to-\infty} \dfrac{\cancel{\frac{1}{x}}^{\,0} - \cancel{2/x^2}^{\,0}}{1 + 4/x^2} = \dfrac{0}{1} = \boxed{0}$

   plug in $x = 1000$
   guess: ∞

   (c) $\lim_{x\to\infty} \dfrac{x^2+3x}{\sqrt{x}+1}$

   Method 1: Divide by $x^{1/2}$

   $\lim_{x\to\infty} \dfrac{\frac{x^2}{x^{1/2}} + \frac{3x}{x^{1/2}}}{1 + \cancel{\frac{1}{x^{1/2}}}^{\,0}} = \lim_{x\to\infty} \dfrac{x^{3/2} + 3x^{1/2}}{1 + 0} = \text{“}\dfrac{\infty}{1}\text{”} = \infty$

   Method 2: Divide by x^2

   $\lim_{x\to\infty} \dfrac{\frac{x^2}{x^2} + \frac{3x}{x^2}}{\frac{x^{1/2}}{x^2} + \frac{1}{x^2}} = \lim_{x\to\infty} \dfrac{1 + \frac{3}{x}}{\frac{1}{x^{3/2}} + \frac{1}{x^2}} = \dfrac{1+0}{0+0} = \text{“}\dfrac{1}{0}\text{”} = \boxed{\infty}$

2. Let $f(x) = -3x^n + 2x^{n-1} + 3x + 1$ where n is a whole number greater that 2. What is $\lim\limits_{x\to\infty} f(x)$ and $\lim\limits_{x\to-\infty} f(x)$. Explain your reasoning. Your answer will depend on n.

$$f(x) = x^n\left(-3 + \frac{2}{x} + \frac{3}{x^{n-1}} + \frac{1}{x^n}\right)$$

So $\lim\limits_{x\to\pm\infty} f(x) = -3x^n$. When n is even $\begin{cases} \lim\limits_{x\to\infty} f(x) = -\infty \\ \lim\limits_{x\to-\infty} f(x) = -\infty \end{cases}$

when n is odd

$\lim\limits_{x\to\infty} f(x) = -\infty$, $\lim\limits_{x\to-\infty} f(x) = \infty$

3. Let $f(x) = \dfrac{2x^2 + 3x - 5}{x^2 + x - 6}$.

(a) Are there horizontal asymptotes? Clearly explain your reasoning.

$$\lim_{x \to \pm\infty} f(x) \overset{\text{divide by } x^2}{=} \lim_{x \to \pm\infty} \frac{2 + 3/x - 5/x^2}{1 + 1/x - 6/x^2}$$

$$= 2.$$

So $y = 2$ is HA

(b) Recall that we have a vertical asymptote at $x = a$ if $\lim_{x \to a} f(x) = \pm\infty$. Are there any vertical asymptotes of this function?

We get $\lim_{x \to a} f(x) = \pm\infty$ when denominator $= 0$

$x^2 + x - 6 = 0 \Rightarrow (x+3)(x-2) = 0 \Rightarrow x = -3, x = 2$

$\lim_{x \to -3} f(x) = \dfrac{18 - 9 - 5}{0} = \dfrac{4}{0} = \pm\infty$, $\lim_{x \to 2} f(x) = \dfrac{8 + 6 - 5}{0} = \dfrac{9}{0} = \pm\infty$

So $x = -3$, $x = 2$ are vertical asymptotes

(c) Use the first derivative to determine when this function is increasing and when it is decreasing.

$$f'(x) = \frac{(x^2+x-6)(4x+3) - (2x^2+3x-5)(2x+1)}{(x^2+x-6)^2}$$

$$= \frac{\cancel{4x^3} + 3x^2 + 4x^2 + 3x - 24x - 18 - (\cancel{4x^3} + 2x^2 + 6x^2 + 3x - 10x - 5)}{(x^2+x-6)^2}$$

$$= \frac{-x^2 - 14x - 13}{(x^2+x-6)^2} = \frac{-(x+1)(x+13)}{(x^2+x-6)^2}$$

$f'(x) = 0$ @ $-1, -13$

undefined $2, -3$

f': $- - -$ | $+ + + + + +$ | $- - - -$ | $- - - -$

$-13 \quad -3 \quad -1 \quad 2$

f inc $(-13, -1)$

f dec $(-\infty, -13) \cup (-1, \infty)$

(d) The second derivative of $f(x) = \dfrac{2(x^3 + 21x^2 + 39x + 55)}{(x^2 + x - 6)^3}$. It is equal to 0 at $x \approx -19.1$. Where is the function concave up and concave down? (Hint: a function can change concavity where the second derivative is 0 OR undefined.)

f'' try -20 try -10 try 0 try 10

$- - - -$ | -1 $+ + + +$ | $- - - -$ | $+ + + +$

$-19.1 \quad -3 \quad 2$

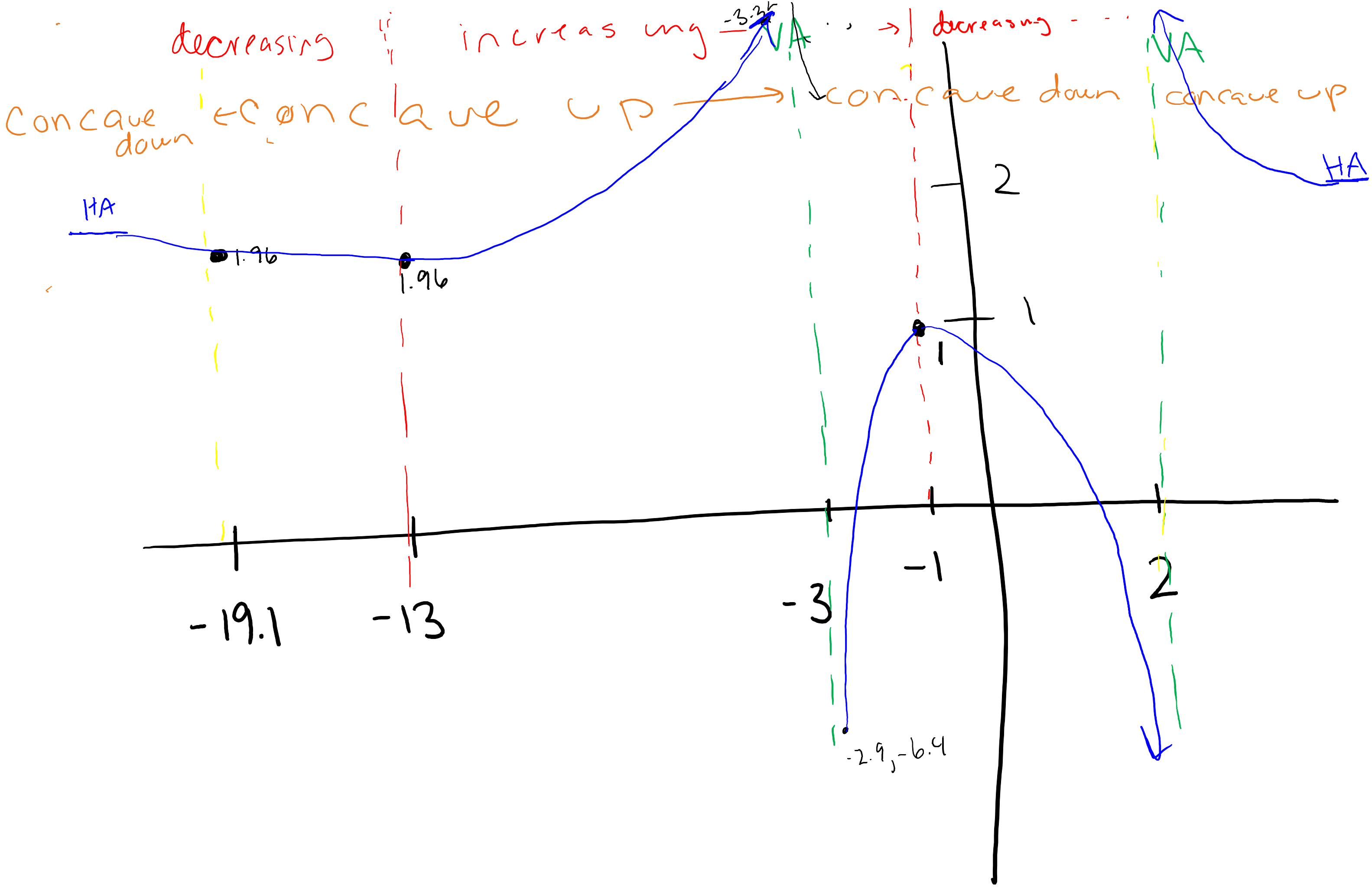
f concave up $(-19.1, -3) \cup (2, \infty)$

f concave down $(-\infty, -19.1) \cup (-3, 2)$

(e) Use the information above to sketch a graph of the function.

see next page

More Practice from Textbook 5.3: You should do as many problems from each set (1-18, 19-32, 33-37), as needed until you are comfortable with these techniques. 19-32 are good practice for application problems.



decreasing
increasing
-3.25
VA
decreasing
VA
concave down
concave up
concave down
concave up
HA
1.96
1.96
2
1
1
HA
-19.1
-13
-3
-1
2
-2.9, -6.4