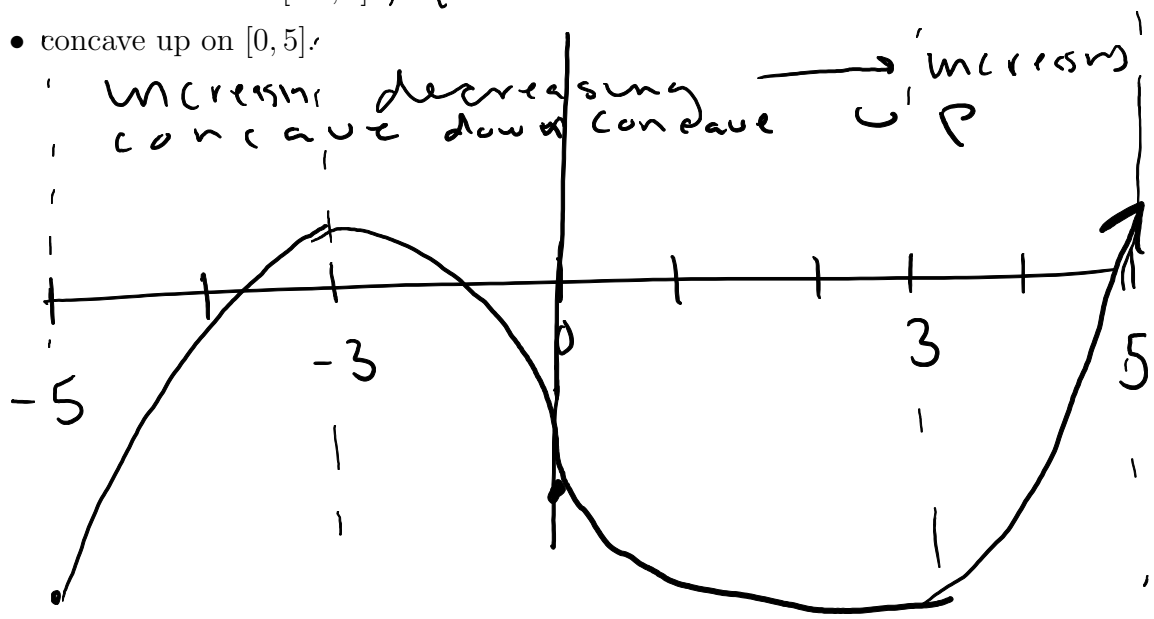
Section 5.2: The Second Derivative
Section Objectives:

- Know how to find the second derivative.
- Know what it means for a functions to be concave up or concave down (both in terms of the first derivative and the graph).
- Know how to use the second derivative to tell if a function is concave up or concave down.
- Know the definition of an inflection point of a function.
- Know how to use the second derivative to determine if a function has a minimum or maximum at a place where the derivative is 0.
- Know how to tell when an economy of scale exists.

Practice Problems

1. Sketch the graph of a function on the domain $[-5, 5]$ which satisfies all the conditions below.
   - increasing on $[-5, -3]$ and $[3, 5]$
   - decreasing on $[-3, 3]$
   - concave down on $[-5, 0]$ ∩
   - concave up on $[0, 5]$.



2. If $f'(4) = 0$ and $f''(4) = 5$, what can we say about f at $x = 4$? Explain your reasoning.

Since $f''(4) = 5 > 0$ f is concave up at $x = 4$ ∪ so critical point ($f'(4) = 0$) must correspond to minimal $x = 4$.

3. Let $f(x) = 3x^4 - 4x^3 + 1$. Find the intervals where $f(x)$ is increasing, decreasing, concave up and concave down. Find all relative extrema and inflection points. Use these to sketch a graph of the function.

$f'(x) = 12x^3 - 12x^2 = 12x^2(x-1)$

$f'(x) = 0$ @ $x = 0, x = 1$

f' $\quad - \; - \; - \; 0 \; - \; - \; - \; - \; 0 \; + \; + \; + \; + \; +$

f $\quad \searrow \; \searrow \; \searrow \; 0 \; \searrow \; \searrow \; \searrow \; \searrow \; 1 \; \nearrow \; \nearrow \; \nearrow \; \nearrow$

f inc $(1, \infty)$ $\quad f$ dec $(-\infty, 1)$ $\quad f(1) = 0$

(no change at $x = 0$), relative min @ $x = 1$ $f(0) = 1$

$f''(x) = 36x^2 - 24x = 12x(3x-2)$ $\quad f(2/3) = .407$

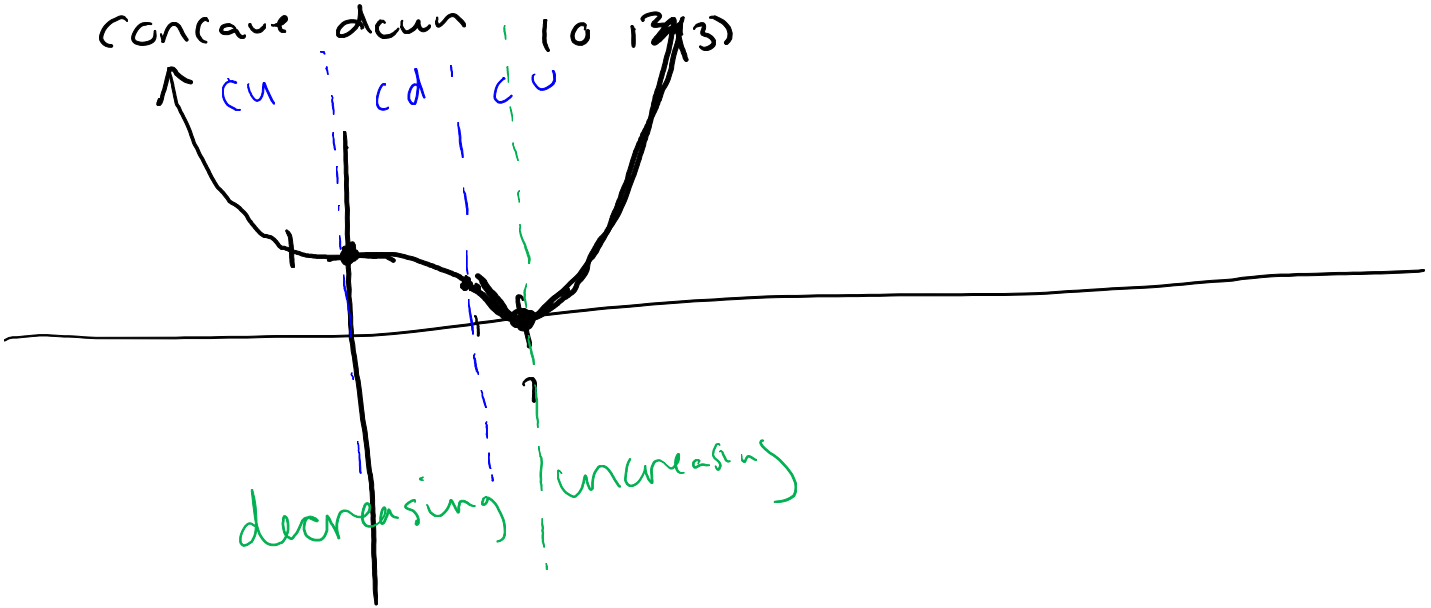
$f''(x) = 0$ @ $x = 0, 2/3$

f'' $\quad + \; + \; + \; - \; - \; - \; - \; + \; + \; + \; + \; +$

f $\quad \cup \; \cup \; \cup \; 0 \; \cap \; \cap \; \cap \; \cap \; 2/3 \; \cup \; \cup \; \cup \; \cup$

f concave up $(-\infty, 0) \cup (2/3, \infty)$

concave down $(0, 2/3)$



4. Let $f(x) = e^{-x^2}$. Find the intervals where $f(x)$ is increasing, decreasing, concave up and concave down. Find all relative extrema and inflection points. Use these to sketch a graph of the function.

$f'(x) = e^{-x^2}(-2x)$

$f'(x) = 0$ when $-2x = 0 \Rightarrow x = 0$

since $e^{-x^2} \neq 0$.

f' $+ + + + + 0 - - - - - -$

f ↗ ↗ ↗ ↗ 0 ↘ ↘ ↘ ↘ ↘

f inc $(-\infty, 0)$

f dec $(0, \infty)$

relative max @ 0, $f(0) = 1$

$f''(x) = e^{-x^2}(-2) + e^{-x^2}(-2x)(-2x)$ (product rule)

$= e^{-x^2}(4x^2 - 2)$

$f''(x) = 0$ when $4x^2 - 2 = 0$

since $e^{-x^2} \neq 0$

$\rightarrow x^2 = 2 \quad x = \pm\sqrt{2}$

f'' $+ + + 0 - - - - - - 0 + + + +$

$\cup \cup -\sqrt{2} \cap \cap \cap \sqrt{2} \cup \cup \cup$

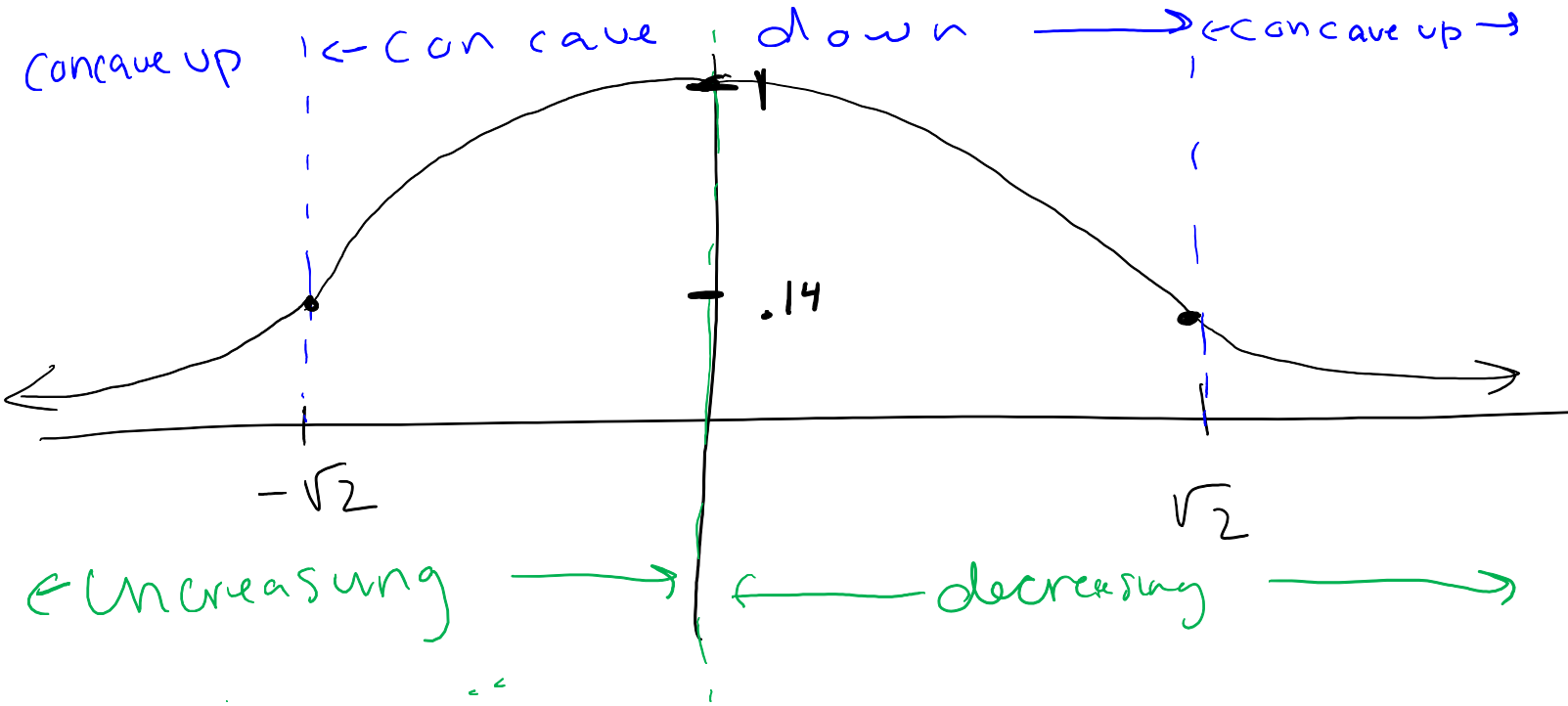
f concave up $(-\infty, -\sqrt{2}) \cup (\sqrt{2}, \infty)$

f concave down $(-\sqrt{2}, \sqrt{2})$

$x = \pm\sqrt{2}$, point of inflection

$f(-\sqrt{2}) = f(\sqrt{2}) \approx .14$

also note: $f(x) > 0$ for all x



5. We have an economy of scale if the marginal cost ($C'(x)$) is decreasing as the number of units produced increases. What does this tell us about $C''(x)$? About $C(x)$?

If $C'(x)$ (marginal cost) is decreasing its derivative $C''(x)$ is negative which means $C(x)$ is concave down when there is an economy of scale.

6. If the cost function of a firm is given by $C(x) = -0.1x^2+2x+5$, is this firm experiencing an economy of scale? Explain your reasoning.

If $C(x) = -.1x^2 + 2x + 5$

then $C'(x) = -.2x + 2$

$C''(x) = -.2 < 0$

so $C''(x) < 0 \Rightarrow C'(x)$ is decreasing

so it is an economy of scale!

More Practice from Textbook 5.2: You should do as many problems from each set (1-6, 7-12, 13-20, 21-24, 25-36, 37-41, 45-62), as needed until you are comfortable with these techniques. 45-62 are good practice for application problems.